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Boundary triples for integral systems on finite intervals

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Presented by M. M. Malamud

Abstract. Let P, Q, and W be real functions of bounded variation on $[0, l]$, and let W be nondecreasing. The integral system

$$J\vec{f}(x) - J\vec{a} = \int_0^x \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix} \vec{f}(t), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{0.1}$$

on a finite compact interval $[0, l]$ was considered in [6]. The maximal and minimal linear relations A_{max} and A_{min} associated with the integral system (0.1) are studied in the Hilbert space $L^2(W)$. It is shown that the linear relation A_{min} is symmetric with deficiency indices $n_\pm(A_{min}) = 2$ and $A_{max} = A_{min}^*$. Boundary triples for A_{max} are constructed, and the the corresponding Weyl functions are calculated.

Keywords. Integral system, boundary triple, symmetric linear relation, deficiency indices, Weyl function.

1. Introduction

This paper focuses on the integral system

$$J\vec{f}(x) - J\vec{a} = \int_0^x dS(t) \cdot \vec{f}(t), \tag{1.1}$$

where J and dS are 2×2 matrices of the form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dS = \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix}, \tag{1.2}$$

$\lambda \in \mathbb{C}$, all the functions P, Q, and W are real of bounded variation on $[0, l]$, and W is nondecreasing on $[0, l]$. Such systems were studied in [2, 3, 6]. System (1.1) contains Sturm–Liouville systems, a Stieltjes string, and a Krein–Feller string [13, 18] as special cases.

With system (1.1), we associate minimal A_{min} and maximal A_{max} linear relations. In contrast to the Sturm–Liouville case, A_{min} and A_{max} may be multivalued. Therefore, we use a term "linear relation" for them (see [1]). It turns out that the linear relation A_{min} is symmetric with deficiency indices $(2, 2)$.

The notions of the boundary triple and Weyl function introduced in [7, 8, 19] and [10], respectively, were proved to be useful in the study of spectral problems and extension-theory problems for symmetric operators (see [11, 12, 14]). Boundary triples for various differential and difference operators were constructed in [4, 10, 11, 14, 19, 21, 22].

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A boundary triple for the linear relation A_{max} will be constructed here, and the corresponding matrix Weyl function will be calculated. In a similar way, some intermediate extensions of the linear relation A_{min} with deficiency indices $(1,1)$ are considered below, and their scalar Weyl functions will be found.

2. Preliminaries

2.1. Linear relations

Let $\mathfrak{H}$ be a Hilbert space. Any linear supspace of $\mathfrak{H}\times\mathfrak{H}$ is called a *linear relation* in $\mathfrak{H}$, [1].

The *domain*, *range*, *kernel*, and *multivalued part* of a linear relation T are defined by the following equalities (see [1,5]):

$$\operatorname{dom} T := \left\{ f : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}, \qquad \operatorname{ran} T := \left\{ g : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}, \tag{2.1}$$

$$\ker T := \left\{ f : \begin{pmatrix} f \\ 0 \end{pmatrix} \in T \right\}, \qquad \operatorname{mul} T := \left\{ g : \begin{pmatrix} 0 \\ g \end{pmatrix} \in T \right\}. \tag{2.2}$$

The adjoint linear relation T^* is defined as

$$T^* := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathfrak{H}\times\mathfrak{H} : \ (v,f)_{\mathfrak{H}} = (u,g)_{\mathfrak{H}} \text{ for some } \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}. \tag{2.3}$$

A linear relation T in $\mathfrak{H}$ is called *closed*, if T is closed as a subspace of $\mathfrak{H}\times\mathfrak{H}$. The set of all closed linear operators (relations) is denoted by $\mathcal{C}(\mathfrak{H})$ ($\widetilde{\mathcal{C}}(\mathfrak{H})$). Identifying a linear operator $T\in\mathcal{C}(\mathfrak{H})$ with its graph, we can consider $\mathcal{C}(\mathfrak{H})$ as a part of $\widetilde{\mathcal{C}}(\mathfrak{H})$.

Definition 2.1. Suppose that T is a linear relation and $\lambda\in\mathbb{C}$. Then

$$T-\lambda I := \left\{ \begin{pmatrix} f \\ g-\lambda f \end{pmatrix} : \ \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\}. \tag{2.4}$$

A point $\lambda\in\mathbb{C}$ such that $\ker(T-\lambda I)=\{0\}$ and $\operatorname{ran}(T-\lambda I)=\mathfrak{H}$ is called a *regular point* of the linear relation T and is denoted by $\lambda\in\rho(T)$.

The *point spectrum* and the *continuous spectrum* of the linear relation T are defined by

$$\sigma_p(T) := \{\lambda\in\mathbb{C} : \ \ker(T-\lambda I)\neq\{0\}\}, \tag{2.5}$$

$$\sigma_c(T) := \{\lambda\in\mathbb{C} : \ \lambda\notin\sigma_p(T), \ \operatorname{ran}(T-\lambda I)\neq\overline{\operatorname{ran}(T-\lambda I)}=\mathfrak{H}\}. \tag{2.6}$$

For $\lambda\in\mathbb{C}_\pm$, let us set $\mathfrak{N}_\lambda(T) := \ker(T^*-\lambda I)$ and

$$\hat{\mathfrak{N}}_\lambda(T) := \left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} : \ f_\lambda\in\mathfrak{N}_\lambda \right\}. \tag{2.7}$$

A linear relation A is called *symmetric*, if $A\subseteq A^*$. The *deficiency indices* of a symmetric linear relation A are defined by

$$n_\pm(A) := \dim\ker(A^*\mp iI). \tag{2.8}$$

2.2. Boundary triples

In the case of densely defined operators, a boundary triple notion was introduced in [7,8,14,19] (in different forms). Following work [21], we give a general definition of a boundary triple for the linear relation T.

Definition 2.2. The tuple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space, Γ_0 and Γ_1 are linear mappings from T to $\mathcal{H}$ is called a *boundary triple* for linear relation T, if the following conditions hold:

(i) generalized Green's identity

$$(g,u)_{\mathfrak{H}} - (f,v)_{\mathfrak{H}} = \left(\Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix}, \Gamma_0 \begin{pmatrix} u \\ v \end{pmatrix}\right)_{\mathcal{H}} - \left(\Gamma_0 \begin{pmatrix} f \\ g \end{pmatrix}, \Gamma_1 \begin{pmatrix} u \\ v \end{pmatrix}\right)_{\mathcal{H}} \tag{2.9}$$

holds for all $\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in T$;

(ii) the mapping $\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \to \mathcal{H} \times \mathcal{H}$ is surjective.

If the linear relation T is adjoint to some symmetric linear relation A, then there exists a boundary triple for T, if the deficiency indices of A coincide ($n_+(A) = n_-(A)$), see [11, 19, 21].

An extension $\widetilde{A}$ of the symmetric linear relation A is called *proper*, if $A \subsetneq \widetilde{A} \subsetneq A^*$. The class of all proper extensions of the linear relation A completed with relations A and A^* is denoted by $\operatorname{Ext}(A)$. Denote also

$$A_\Theta := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in A^* : \ \Gamma \begin{pmatrix} f \\ g \end{pmatrix} \in \Theta \right\}. \tag{2.10}$$

Proposition 2.3. [11] *Let A be a symmetric linear relation, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the adjoint linear relation A^*. Then the mapping $\Gamma : \widetilde{A} = A_\Theta \to \Theta = \Gamma\widetilde{A}$ is a one-to-one mapping from $\operatorname{Ext}(A)$ to $\widetilde{\mathcal{C}}(\mathfrak{H})$. Note also that A_Θ is self-adjoint, iff the linear relation Θ is self-adjoint.*

In particular, the linear relations

$$A_0 := \ker \Gamma_0, \qquad A_1 := \ker \Gamma_1 \tag{2.11}$$

are disjoint, i.e., $A_0 \cap A_1 = A$, and they are self-adjoint extensions of the symmetric linear relation A (see [11]).

Suppose that A is adjoint for the linear relation T from Definition 2.2. The conditions ensuring the symmetry of A are provided by the next theorem (in the case of single-valued linear operator T, the corresponding theorem was proved in [12]).

Theorem 2.4. [12] *Let T be a linear relation in the Hilbert space $\mathfrak{H}$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be its boundary triple such that $n := \dim \mathcal{H} < \infty$ and $A = \ker \Gamma$. If the conditions*

(i) $\operatorname{ran} T = \mathfrak{H}$;

(ii) $\dim \ker T = n$, *and* $\ker A = \{0\}$

hold, then the linear relations A and T are closed, $T = A^*$, and $n_+(A) = n_-(A) = n$.

Definition 2.5. [10, 11] Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the linear relation A^*. The operator-valued functions $M(\cdot)$ and $\gamma(\cdot)$ defined by

$$M(\lambda)\Gamma_0\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0\hat{f}_\lambda = f_\lambda, \quad \hat{f}_\lambda \in \hat{\mathfrak{N}}_\lambda, \quad \lambda \in \rho(A_0) \tag{2.12}$$

are called the Weyl function and the γ-field of the symmetric linear relation A with respect to the boundary triple Π.

Definition 2.6. An operator-valued function $F : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathcal{B}(\mathcal{H})$ is said to belong to the class $R[\mathcal{H}]$, if the following conditions hold:

(i) F is holomorphic in $\mathbb{C}_+ \cup \mathbb{C}_-$;

(ii) $\operatorname{Im} F(\lambda) \geq 0$ as $\lambda \in \mathbb{C}_+$;

(iii) $F(\overline{\lambda}) = F^*(\lambda), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

If $\mathcal{H} = \mathbb{C}$ then $R[\mathcal{H}]$ is denoted by R.

It is known that the Weyl function $M(\lambda)$ of a linear relation A from Definition 2.5 belongs to the class $R[\mathcal{H}]$. The next proposition gives a description of the spectrum of a linear $\widetilde{A} \in \operatorname{Ext}(A)$.

Proposition 2.7. [11] *Let A be a symmetric linear relation in $\mathfrak{H}$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*, let $M(\lambda)$ be the corresponding Weyl function of A, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, and let $\lambda \in \rho(A_0)$. Then*

(i) $\lambda \in \rho(\widetilde{A}_\Theta) \Longleftrightarrow 0 \in \rho(\Theta - M(\lambda))$;

(ii) $\lambda \in \sigma_p(\widetilde{A}_\Theta) \Longleftrightarrow 0 \in \sigma_p(\Theta - M(\lambda))$.

2.3. Integral systems

On the compact segment $[0, l]$, we now consider the integral system

$$J\vec{f}(x) - J\vec{a}(x) = \int\limits_0^x dS(t) \cdot \vec{f}(t), \tag{2.13}$$

where $\vec{f}$ is an $n \times 1$ complex vector, $\vec{a}$ is a fixed complex vector-valued function of bounded variation, dS is a finite $n \times n$ measure, and J is a constant $n \times n$ matrix such that $J^* = -J$.

Definition 2.8. We say that a vector-valued function $\vec{f}$ is a solution of the integral system (2.13), if (each component of) $\vec{f}$ is of bounded variation, and equality (2.13) holds for every point of $[0, l]$.

It is easy to see that if, for some vector-valued function $\vec{f}$, the right-hand side of equality (2.13) exists for all $x \in [0, l]$, then it is of bounded variation on $[0, l]$. Therefore, the inclusion $\vec{f} \in BV[0, l]$ is necessery for (2.13). The same condition is also sufficient for the existence of the integral on the right-hand side of (2.13) (as a Lebesgue–Stietjes integral).

In the general case, the measure dS is not supposed to be absolutely continuous and may have mass points on $[0, l]$. Therefore, in equality (2.13) and in the following, we should understand $\int_a^b f d\mu$ as the Lebesgue–Stieltjes integral $\int f\chi_{[a,b)} d\mu$, where $\chi_{[a,b)}$ is the characteristic function of the half-open interval. Under this conventions, the integrals as functions of their limits of integration are left-continuous.

The following theorem was proved in [6].

Theorem 2.9. [6] *For any left-continuous vector-function* $\vec{a}(x) \in BV[0,l]$, *there exists a unique solution of* (2.13).

In what follows, the integration-by-parts formula will be used in the following form (see [15]). If u is a left-continuous function of bounded variation, then we denote, by u_+, the right-continuous function that coincides with u at every continuity point. If v is another left-continuous function of bounded variation, then the following equality holds:

$$\int_y^x v du = v(x)u(x) - v(y)u(y) - \int_y^x u_+ dv. \tag{2.14}$$

Now, we suppose that $n = 2$, the matrices J and dS have the form

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad dS = \begin{pmatrix} \lambda dW - dQ & 0 \\ 0 & dP \end{pmatrix}, \tag{2.15}$$

where λ is a complex parameter, P, Q, and W are functions of bounded variation, left-continuous on $[0,l]$, and satisfy the condition

$$P(0) = Q(0) = W(0) = 0, \tag{2.16}$$

and W is nondecreasing. We assume that the functions P, Q, and W are defined on the whole real line, and their values on the intervals $(-\infty, 0]$ and $[l, +\infty)$ are constant.

In the remaining part of this paper, the attention will be restricted to considering (2.13) in the case where the matrices J and dS have the form (2.15).

Everywhere in the following, we use

Assumption 2.10. *The functions* Q *and* W *have no common discontinuities with* P.

3. Green's identity and linear relation A_{max}

3.1. Green's identity

Let $\mathcal{L}^2(W)$ be an inner product space, which consists of complex-valued functions f such that

$$\int_0^l |f(t)|^2 dW(t) < \infty. \tag{3.1}$$

The inner product in $\mathcal{L}^2(W)$ is defined by

$$(f,g)_W = \int_0^l f(t)\overline{g(t)} dW(t). \tag{3.2}$$

By $L^2(W)$, we denote the corresponding quotient space, which consists of equivalence classes with respect to the measure dW. To avoid confusion, we will denote elements of the space $L^2(W)$ by gothic letters $\mathfrak{f}$, $\mathfrak{g}$, etc.

Let us consider the inhomogeneous system

$$J \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} \Bigg|_0^x = \int_0^x \begin{pmatrix} -dQ(t) & 0 \\ 0 & dP(t) \end{pmatrix} \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} + \int_0^x \begin{pmatrix} dW(t) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{3.3}$$

Definition 3.1. A pair $\{\vec{f}, g\}$ that consists of a vector-function $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and a scalar function g is said to satisfy system (3.3) (or that $\vec{f}$ is a solution of this system with fixed g), if the following conditions hold:

(i) $g \in \mathcal{L}^2(W)$;

(ii) $\vec{f} \in BV[0, l]$;

(iii) equality (3.3) holds for every $x \in [0, l]$.

Remark. It is clear that the condition $\vec{f} \in BV[0, l]$ is automatically satisfied, as equality (3.3) holds. In this case, it follows from $\vec{f} \in BV[0, l]$ that $f \in \mathcal{L}^2(W)$.

The componentwise rewriting of system (3.3) gives

$$\begin{cases} f(x) - f(0) = \int\limits_0^x f^{[1]}(t) dP(t), \\ f^{[1]}(x) - f^{[1]}(0) = \int\limits_0^x \left(f(t) dQ(t) - g(t) dW(t) \right). \end{cases} \tag{3.4}$$

Theorem 3.2 (The first Green's identity)**.** *Let Assumption 2.10 holds and pairs* $\left\{\vec{f}, g\right\}$, $\{\vec{u}, v\}$ *satisfy system* (3.3) *(see Definition 3.1). Then, for any* α *and* $\beta \in [0, l]$, *the following equality holds:*

$$\int_\alpha^\beta gu\ dW = \int_\alpha^\beta fu\ dQ + \int_\alpha^\beta f^{[1]} u^{[1]}\ dP - f^{[1]} u \Big|_\alpha^\beta. \tag{3.5}$$

Proof. From (3.4), we have

$$du = u^{[1]} dP, \quad df^{[1]} = f dQ - g dW. \tag{3.6}$$

It follows from Assumption 2.10 that functions u and $f^{[1]}$ have no common discontinuities. Consider the measure $d\left(f^{[1]} u\right)$. Then

$$d\left(f^{[1]} u\right) = df^{[1]} u + f^{[1]} du = fu\ dQ + f^{[1]} u^{[1]}\ dP - gu\ dW, \tag{3.7}$$

hence

$$gu\ dW = fu\ dQ + f^{[1]} u^{[1]}\ dP - d\left(f^{[1]} u\right). \tag{3.8}$$

To conclude the proof it remains to note that function $f^{[1]} u$ is left-continuous and to integrate equality (3.8) over $[\alpha, \beta)$. □

For a pair of vector-valued functions $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}$ we define the generalized Wronskian by

$$\left[\vec{f}, \vec{u}\right] := \left(f u^{[1]} - f^{[1]} u \right). \tag{3.9}$$

Theorem 3.3. *Suppose Assumption 2.10 holds and pairs* $\left\{\vec{f}, g\right\}$, $\{\vec{u}, v\}$ *satisfy system* (3.3). *Then for any* $\alpha, \beta \in [0, l]$ *the next equality holds*

$$\int_\alpha^\beta (gu - fv)\ dW = \left[\vec{f}, \vec{u}\right] \Big|_\alpha^\beta. \tag{3.10}$$

Proof. Application of Theorem 3.2 gives

$$gu\ dW = fu\ dQ + f^{[1]}u^{[1]}\ dP - d\left(f^{[1]}u\right), \tag{3.11}$$

$$fv\ dW = fu\ dQ + f^{[1]}u^{[1]}\ dP - d\left(fu^{[1]}\right). \tag{3.12}$$

Subtraction of (3.12) from (3.11) proves the statement. □

Corollary 3.4 (The second Green's identity). *For any two pairs* $\left\{\vec{f}, g\right\}$ *and* $\{\vec{u}, v\}$ *satisfying* (3.3) *the generalized Green's identity holds*

$$(g, u)_W - (f, v)_W = \left(f^{[1]}\overline{u}|_0 - f^{[1]}\overline{u}|_l\right) - \left(f\overline{u^{[1]}}|_0 - f\overline{u^{[1]}}|_l\right). \tag{3.13}$$

3.2. Linear relation A_{max}

Definition 3.5. We shall say that a pair of classes $\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix} \in L^2(W) \times L^2(W)$ belongs to the linear relation A_{max} if there exist functions $f, f^{[1]}$ and g such that

(i) the pair $\left\{\vec{f}, g\right\}$, where $\vec{f} = \begin{pmatrix} f \\ f^{[1]}\end{pmatrix}$, satisfies (3.3) (in the sense of Definition 3.1);

(ii) $f \in \mathfrak{f}$, $g \in \mathfrak{g}$.

In the succeeding we require the following

Assumption 3.6. *For any* $a, b, a_1, b_1 \in \mathbb{C}$ *there exists a pair* $\left\{\vec{f}, g\right\}$ *satisfying* (3.3) *such that*

$$f(0) = a, \quad f^{[1]}(0) = a_1, \quad f(l) = b, \quad f^{[1]}(l) = b_1. \tag{3.14}$$

In particular, if $dQ \equiv 0$ then a sufficient condition for Assumption 3.6 to hold is the next

Proposition 3.7. *Suppose* $dQ \equiv 0$. *If there exist closed on the left and disjoint intervals* i_1 *and* i_2 *on* $[0, l]$ *such that*

$$\dim L^2(i_j, W) > 0 \quad (j \in \{1, 2\}), \tag{3.15}$$

$$\frac{1}{dW(i_2)} \int_{i_2} P(t)dW(t) > \frac{1}{dW(i_1)} \int_{i_1} P(t)dW(t), \tag{3.16}$$

then Assumption 3.6 holds.

Proof. Let $(a\ b\ a_1\ b_1)^T$ be an arbitrary vector from $\mathbb{C}^4$. It follows from condition (3.15) that there exist functions u_j that are equal to 1 on the interval i_j and to zero on its complement, and $\|u_j\|_W = dW(i_j) \neq 0$ $(j \in \{1, 2\})$.

We set $g = c_1u_1 + c_2u_2$, where c_1 and c_2 are some constants from $\mathbb{C}$. We define a vector-function $\vec{f}$ by the system

$$\begin{cases} f(x) = a + \int\limits_0^x f^{[1]}(t)dP(t), \\ f^{[1]}(x) = a_1 - \int\limits_0^x g(t)dW(t). \end{cases} \tag{3.17}$$

It is clear that, for any c_1 and $c_2 \in \mathbb{C}$, we have $g \in \mathcal{L}^2(W)$. Further, it follows from system (3.17) that the vector-function $\vec{f}$ is of bounded variation on $[0, l]$, and $\vec{f}(0) = (a\ a_1)^T$, i.e., the pair $\left\{\vec{f}, g\right\}$ satisfies system (3.3) with the initial conditions given in advance.

Let us show now that the constants c_1 and c_2 can be chosen so that the equality $\vec{f}(l) = (b\ b_1)^T$ holds. It is true, iff there exists a solution of the system (with respect to c_1 and c_2)

$$\begin{cases} c_1 dW(i_1) + c_2 dW(i_2) = a_1 - b_1, \\ c_1 \int\limits_0^l dP(t) \int\limits_0^t u_1(s)dW(s) + c_2 \int\limits_0^l dP(t) \int\limits_0^t u_2(s)dW(s) = \\ \qquad\qquad\qquad\qquad\qquad a - b + a_1 P(l). \end{cases} \tag{3.18}$$

By Assumption 2.10, the functions P and W have no common discontinuities. So, using integration by parts formula (2.14), we get

$$\int\limits_0^l dP(t) \int\limits_0^t u_j(s)dW(s) = P(l)dW(i_j) - \int\limits_0^l P(t)u_j(t)dW(t) \tag{3.19}$$

where $j \in \{1, 2\}$. Multiplying the first equation of system (3.18) by $P(l)$, subtracting it from the second one, and combining the obtained equation with (3.19), we have a system (with respect to c_1 and c_2), whose determinant

$$\begin{vmatrix} dW(i_1) & dW(i_2) \\ \int_{i_1} P(t)dW(t) & \int_{i_2} P(t)dW(t) \end{vmatrix} \tag{3.20}$$

is strictly positive due to (3.16). This ensures the solvability of system (3.18). □

Theorem 3.8. *Let Assumption 2.10 and Assumption 3.6 be satisfied, and let the mappings* $\Gamma_0, \Gamma_1 : A_{max} \to \mathbb{C}^2$ *be defined by*

$$\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f(0) \\ f(l) \end{pmatrix}, \quad \Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} := \begin{pmatrix} f^{[1]}(0) \\ -f^{[1]}(l) \end{pmatrix}, \tag{3.21}$$

where the pair $\left\{\vec{f}, g\right\}$ *satisfies system* (3.3), $f \in \mathfrak{f}$, $g \in \mathfrak{g}$. *Then*

(i) the mappings $\Gamma_0,\ \Gamma_1$ *are well-defined;*

(ii) the tuple $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ *is a boundary triple for the linear relation* A_{max}.

Proof. (i) Let us show firstly that the mappings Γ_0 and Γ_1 from (3.21) are independent of the choice of f and g from the classes $\mathfrak{f}$ and $\mathfrak{g}$, respectively. It is clear that if a pair $\left\{\vec{f}, g_1\right\}$ satisfies system (3.3), then a pair $\left\{\vec{f}, g_2\right\}$ also satisfies (3.3), if g_1 and g_2 are equivalet in the measure dW. This means that the values of $\Gamma_0,\ \Gamma_1$ are independent of the choice of $g \in \mathfrak{g}$.

Further, we prove that the values of the mappings Γ_0, Γ_1 are independent of choosing an instance f from the class $\mathfrak{f}$. Let the pairs $\left\{\vec{f_1}, g\right\}$ and $\left\{\vec{f_2}, g\right\}$ satisfy system (3.3) such that $f_1, f_2 \in \mathfrak{f}$. The application of Green's identity in the form (3.10) to both pairs on $[0, l]$ gives us two equalities. Subtracting one equality from another one gives

$$0 = \int\limits_0^l (f_2 - f_1)\overline{v}dW = \left[\vec{f_1} - \vec{f_2}, \overline{\vec{u}}\right]\Big|_0^l. \tag{3.22}$$

By Assumption 3.6, the pair of functions $\{\vec{u}, v\}$ satisfying system (3.3) can be chosen so that $u(0)$, $u(l)$, $u^{[1]}(0)$, and $u^{[1]}(l)$ may be arbitrary from $\mathbb{C}$. This means that we have

$$f_1(0) = f_2(0), \qquad f_1(l) = f_2(l), \tag{3.23}$$

$$f_1^{[1]}(0) = f_2^{[1]}(0), \qquad f_1^{[1]}(l) = f_2^{[1]}(l) \tag{3.24}$$

which proves that the mappings Γ_0 and Γ_1 are single-valued.

(ii) It follows directly from Corollary 3.4 and Assumption 3.6 that the requirements of Definition 2.2 are satisfied. □

Remark 3.9. Evidently, if Assumption 3.6 does not hold, then the objects Γ_0 and Γ_1 defined by (3.21) are not operators in the general case, but linear relations in $L^2(W)^2 \times \mathbb{C}^2$. Such boundary triples were considered in [9].

It is also possible that if Assumption 3.6 does not hold, then the mapping $\Gamma = (\Gamma_0\ \Gamma_1)^T$ is not surjective. This happens, for example, if $dQ = 0$, $dP = dx$, and W is piecewise with a single jump.

In the case of $dQ \equiv 0$, system (3.4) can be rewritten as follows:

$$f(x) = f(0) + f^{[1]}(0)P(x) - \int_0^x \left\{ \int_0^t g(s)dW(s) \right\} dP(t). \tag{3.25}$$

The function $G(t) := \int_0^t g(s)dW(s)$ is of bounded variation on $[0, l]$, and the set of its jumps is a subset of the set of jumps of the function W. Hence, the functions G and P have no common discontinuities. The application of the integration-by-parts formula (2.14) to equality (3.25) gives us (cf. [17, p. 650, equality (1.1)])

$$f(x) = f(0) + f^{[1]}(0)P(x) - \int_0^x \{P(x) - P(t)\}\, g(t)dW(t). \tag{3.26}$$

This leads to the following assertion.

Proposition 3.10. *Suppose that Assumption 2.10 holds and $dQ \equiv 0$. Then the kernel of the linear relation $A_{max} \subset L^2(W)^2$ is two-dimensional, if the function P is not equivalent to a constant in $L^2(W)$ and is one-dimensional otherwise.*

Proof. Let g be the zero element of $L^2(W)$. Then equality (3.26) takes the form

$$f(x) = f(0) + f^{[1]}(0)P(x), \tag{3.27}$$

which is equivalent to $f \in \operatorname{span}\{1, P\}$. □

Remark. In the proof of Theorem 3.12, it will be shown that the kernel of the linear relation A_{max} is always two-dimensional, if, in addition, Assumption 3.6 holds.

Definition 3.11. We say that an element $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix}$ of the linear relation A_{max} belongs to the linear relation A_{min}, if

$$f(0) = f^{[1]}(0) = f(l) = f^{[1]}(l) = 0. \tag{3.28}$$

It follows from equality (3.13) that the linear relation A_{min} is symmetric.

Theorem 3.12. *The linear relations A_{min} and A_{max} are closed, $A^*_{min} = A_{max}$, and the deficiency indices of A_{min} are $(2, 2)$.*

Proof. We now verify that, for the linear relations A_{min} and A_{max}, the conditions of Theorem 2.4 are satisfied. It follows directly from Theorem 2.9 that $\operatorname{ran} A_{max} = L^2(W)$. Let $\mathfrak{g}$ be an arbitrary class from $L^2(W)$, and let g be some instance of $\mathfrak{g}$. Then, (for any fixed initial value) by Theorem 2.9, there exists a vector-function $\vec{f} = \begin{pmatrix} f \\ f^{[1]} \end{pmatrix}$ such that the pair $\left\{\vec{f}, g\right\}$ satisfies system (3.3) and, as a consequence, $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max}$.

Further, let us show that $\dim \ker A_{max} = 2$. By Theorem 2.9, if $g = 0$, then, for any complex numbers a and a_1, there exists a unique vector-function $\vec{f}$ such that $f(0) = a, f^{[1]}(0) = a_1$, and $\mathfrak{f} \in \ker A_{max}$, where $\mathfrak{f}$ is the class from $L^2(W)$ generated by f. If Assumption 3.6 holds, then, similarly to the proof of Theorem 3.8, we get that $\dim \ker A_{max}$ is isomorphic to $\mathbb{C}^2$. By the same argument, we get $\ker A_{min} = \{0\}$. Now, the statement of this theorem follows from Theorem 2.4. □

Theorem 3.13. [25] *The set of all self-adjoint extensions of the linear relation A_{min} is described by the boundary conditions*

$$\widetilde{A} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : C\Gamma_0 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} + D\Gamma_1 \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = 0 \right\} \tag{3.29}$$

where C, D are complex-valued 2×2 matrices such that

$$\det(CC^* + DD^*) \neq 0, \quad CD^* = DC^*. \tag{3.30}$$

In particular, the linear relations A_0 and A_1 defined by equalities (2.11) are self-adjoint extensions of the linear relation A_{min}. The extensions A_0 and A_1 corresponding to the boundary triple (3.21) coincide with the Dirichlet extension and the Neumann extension

$$A_D := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = 0 \right\}, \tag{3.31}$$

$$A_N := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(0) = f^{[1]}(l) = 0 \right\}, \tag{3.32}$$

respectively.

3.3. Functions $c(x, \lambda)$ and $s(x, \lambda)$. Weyl function of linear relation A_{max}

Let Assumption 2.10 and Assumption 3.6 hold. It follows from Theorem 2.9 that, for each fixed $\lambda \in \mathbb{C}$, there exist unique vector-functions $\vec{c}(x, \lambda)$ and $\vec{s}(x, \lambda)$ satisfying the initial conditions

$$\begin{aligned} c(0, \lambda) = 1, \quad c^{[1]}(0, \lambda) = 0, \\ s(0, \lambda) = 0, \quad s^{[1]}(0, \lambda) = 1, \end{aligned} \tag{3.33}$$

such that the pairs $\{\vec{c}, \lambda c\}$ and $\{\vec{s}, \lambda s\}$ satisfy system (3.3). Here, we have inclusions $c,\ s \in \mathcal{L}^2(W)$. Let $\mathfrak{c}(\lambda)$ and $\mathfrak{s}(\lambda)$ be classes from $L^2(W)$ generated by $c(x, \lambda)$ and $s(x, \lambda)$, respectively. Then

$$\begin{pmatrix} \mathfrak{c}(\lambda) \\ \lambda\mathfrak{c}(\lambda) \end{pmatrix}, \begin{pmatrix} \mathfrak{s}(\lambda) \\ \lambda\mathfrak{s}(\lambda) \end{pmatrix} \in A_{max}. \tag{3.34}$$

It is known (see [6]) that the functions $c(x,\lambda)$ and $s(x,\lambda)$ are entire in λ of order not greater that 1/2.

By conditions (3.33), the functions c and s are linearly independent, and it follows from Assumption 3.6 that the classes $\mathfrak{c}(\lambda)$ and $\mathfrak{s}(\lambda)$ are linearly independent as well. Any element $\mathfrak{f}_\lambda$ from the defect subspace $\mathfrak{N}_\lambda$ can be represented as

$$\mathfrak{f}_\lambda = a_1\mathfrak{c}(\lambda) + a_2\mathfrak{s}(\lambda), \quad a_1, a_2 \in \mathbb{C}. \tag{3.35}$$

Theorem 3.14. *The generalized Wronskian of the functions $\vec{c}(x,\lambda)$ and $\vec{s}(x,\lambda)$ is a constant:*

$$[\vec{c},\vec{s}] = c(x,\lambda)s^{[1]}(x,\lambda) - c^{[1]}(x,\lambda)s(x,\lambda) = 1, \quad x \in [0,l]. \tag{3.36}$$

Proof. Note that both pairs $\begin{pmatrix}\mathfrak{s}\\ \lambda\mathfrak{s}\end{pmatrix}$ and $\begin{pmatrix}\bar{\mathfrak{s}}\\ \bar{\lambda}\bar{\mathfrak{s}}\end{pmatrix}$ belong or do not belong to the linear relation A_{max} simultaneously. The application of Green's identity in the form (3.10) to the pairs $\begin{pmatrix}\mathfrak{c}\\ \lambda\mathfrak{c}\end{pmatrix}$ and $\begin{pmatrix}\bar{\mathfrak{s}}\\ \bar{\lambda}\bar{\mathfrak{s}}\end{pmatrix}$ gives

$$[\vec{c}(t,\lambda),\vec{s}(t,\lambda)]|_0^x = 0. \tag{3.37}$$

□

Theorem 3.15. *The Weyl function and the γ-field of the linear relation A_{max} corresponding to the boundary triple $\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ from (3.21) have the forms*

$$M(\lambda) = \frac{-1}{s(l,\lambda)}\begin{pmatrix} c(l,\lambda) & -1 \\ -1 & s^{[1]}(l,\lambda)\end{pmatrix}, \tag{3.38}$$

$$\gamma(\lambda) = \frac{1}{s(l,\lambda)}\begin{pmatrix}\mathfrak{c}(\lambda)s(l,\lambda) - c(l,\lambda)\mathfrak{s}(\lambda) & \mathfrak{s}(\lambda)\end{pmatrix}. \tag{3.39}$$

Proof. Let $\mathfrak{f}_\lambda = a_1\mathfrak{c}(\lambda) + a_2\mathfrak{s}(\lambda)$. Then

$$\begin{aligned}\Gamma_0\begin{pmatrix}\mathfrak{f}_\lambda\\ \lambda\mathfrak{f}_\lambda\end{pmatrix} &= \begin{pmatrix}1 & 0\\ c(l,\lambda) & s(l,\lambda)\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} =: Y_0\begin{pmatrix}a_1\\ a_2\end{pmatrix},\\ \Gamma_1\begin{pmatrix}\mathfrak{f}_\lambda\\ \lambda\mathfrak{f}_\lambda\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -c^{[1]}(l,\lambda) & -s^{[1]}(l,\lambda)\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} =: Y_1\begin{pmatrix}a_1\\ a_2\end{pmatrix}.\end{aligned} \tag{3.40}$$

It follows from Definition 2.5 of the Weyl function and equality (3.36) that

$$\begin{aligned}M(\lambda) = Y_1Y_0^{-1} &= \frac{-1}{s(l,\lambda)}\begin{pmatrix} c(l,\lambda) & -1\\ c^{[1]}(l,\lambda)s(l,\lambda) - c(l,\lambda)s^{[1]}(l,\lambda) & s^{[1]}(l,\lambda)\end{pmatrix}\\ &= \frac{-1}{s(l,\lambda)}\begin{pmatrix}c(l,\lambda) & -1\\ -1 & s^{[1]}(l,\lambda)\end{pmatrix}.\end{aligned} \tag{3.41}$$

Finally, by definition of the γ-field, we have

$$\begin{aligned}\gamma(\lambda) &= \begin{pmatrix}\mathfrak{c}(\lambda) & \mathfrak{s}(\lambda)\end{pmatrix} Y_0^{-1}\\ &= \frac{1}{s(l,\lambda)}\begin{pmatrix}\mathfrak{c}(\lambda)s(l,\lambda) - c(l,\lambda)\mathfrak{s}(\lambda) & \mathfrak{s}(\lambda)\end{pmatrix}.\end{aligned} \tag{3.42}$$

□

4. Weyl functions of intermediate extensions of linear relation A_{min}

In this section, the boundary triples and the corresponding Weyl functions for intermediate extensions of the linear relation A_{min} are constructed.

Definition 4.1. Let us set

$$A_{D0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_D : f^{[1]}(0) = 0 \right\}, \qquad A_{Dl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_D : f^{[1]}(l) = 0 \right\},$$

$$A_{N0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_N : f(0) = 0 \right\}, \qquad A_{Nl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_N : f(l) = 0 \right\}.$$

It follows from Definition 4.1, (3.31), and (3.32) that

$$A_{D0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = f^{[1]}(0) = 0 \right\}, \tag{4.1}$$

$$A_{Dl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f(l) = f^{[1]}(l) = 0 \right\}, \tag{4.2}$$

$$A_{N0} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(0) = f^{[1]}(0) = f^{[1]}(l) = 0 \right\}, \tag{4.3}$$

$$A_{Nl} := \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(l) = f^{[1]}(0) = f^{[1]}(l) = 0 \right\}. \tag{4.4}$$

Theorem 4.2. *The linear relation A_{D0} is symmetric in $L^2(W)$ with deficiency indices $(1,1)$, and the following conditions hold:*

(i) The adjoint linear relation A_{D0}^* has the form

$$A_{D0}^* = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f(l) = 0 \right\}. \tag{4.5}$$

(ii) The tuple $\{\mathbb{C}, \Gamma_0^{D0}$, and $\Gamma_1^{D0}\}$, where

$$\Gamma_0^{D0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(0), \quad \Gamma_1^{D0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = -f(0), \tag{4.6}$$

is a boundary triple for A_{D0}^*.

(iii) The corresponding Weyl function and the γ-field have the form

$$M_{D0}(\lambda) = \frac{s(l,\lambda)}{c(l,\lambda)}, \quad \gamma_{D0}(\lambda) = \mathfrak{s}(\lambda) - \frac{s(l,\lambda)}{c(l,\lambda)}\mathfrak{c}(\lambda). \tag{4.7}$$

Proof. (i) Suppose that $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{D0}$. By definition, $\begin{pmatrix} \mathfrak{u} \\ \mathfrak{v} \end{pmatrix} \in A_{D0}^*$ holds, iff

$$(\mathfrak{g}, \mathfrak{u})_{L^2(W)} = (\mathfrak{f}, \mathfrak{v})_{L^2(W)}. \tag{4.8}$$

The last equality is equivalent to

$$\left(\Gamma_1\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix},\Gamma_0\begin{pmatrix}\mathfrak{u}\\ \mathfrak{v}\end{pmatrix}\right)_{\mathcal{H}}=\left(\Gamma_0\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix},\Gamma_1\begin{pmatrix}\mathfrak{u}\\ \mathfrak{v}\end{pmatrix}\right)_{\mathcal{H}}. \tag{4.9}$$

Since $f^{[1]}(l)$ is arbitrary, the last equality holds, iff $u(l)=0$.

(ii) Let us show that Green's identity (in the sense of Definition 2.2) holds for the mappings Γ_0^{D0} and Γ_1^{D0}, which are defined on A_{D0}^*. It is clear that $A_{D0}^*\subset A_{max}$. Hence, for any $\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix},\begin{pmatrix}\mathfrak{u}\\ \mathfrak{v}\end{pmatrix}\in A_{D0}^*$, equality (3.13) holds. With regard for (4.5), we have

$$(\mathfrak{g},\mathfrak{u})_{L^2(W)}-(\mathfrak{f},\mathfrak{v})_{L^2(W)}=f^{[1]}(0)\overline{u(0)}-f(0)\overline{u^{[1]}(0)}. \tag{4.10}$$

It remains to verify that the mapping $\Gamma_{D0}=\begin{pmatrix}\Gamma_0^{D0}\\ \Gamma_1^{D0}\end{pmatrix}: A_{D0}^*\to\mathbb{C}\oplus\mathbb{C}$ is surjective, which follows directly from the subjectivity of the mapping Γ on A_{max}.

(iii) The defect subspace of the linear relation A_{D0}^* has the form

$$\mathfrak{N}_\lambda(A_{D0}^*)=\operatorname{span}\{\mathfrak{c}(\lambda)+k\mathfrak{s}(\lambda)\}, \tag{4.11}$$

where the coefficient k is chosen to satisfy $f_\lambda(l)=0$. Further,

$$\Gamma_0^{D0}\hat{f}_\lambda=k=-\frac{c(l,\lambda)}{s(l,\lambda)},\quad \Gamma_1^{D0}\hat{f}_\lambda=-1, \tag{4.12}$$

and, finally,

$$M_{D0}(\lambda)=\frac{s(l,\lambda)}{c(l,\lambda)},\quad \gamma_{D0}(\lambda)=\mathfrak{s}(\lambda)-\frac{s(l,\lambda)}{c(l,\lambda)}\mathfrak{c}(\lambda). \tag{4.13}$$

□

Similar theorems for the extensions A_{Dl}, A_{N0}, and A_{Nl} are given below without proofs.

Theorem 4.3. *The linear relation A_{Dl} is symmetric in $L^2(W)$ with deficiency indices $(1,1)$, and the following conditions hold:*

(i) The adjoint linear relation A_{Dl}^* has the form

$$A_{Dl}^*=\left\{\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix}\in A_{max}: f(0)=0\right\}. \tag{4.14}$$

(ii) The tuple $\{\mathbb{C},\Gamma_0^{Dl},\Gamma_1^{Dl}\}$, where

$$\Gamma_0^{Dl}\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix}=f^{[1]}(l),\quad \Gamma_1^{Dl}\begin{pmatrix}\mathfrak{f}\\ \mathfrak{g}\end{pmatrix}=f(l), \tag{4.15}$$

is a boundary triple for A_{Dl}^*.

(iii) The corresponding Weyl function and the γ-field have the form

$$M_{Dl}(\lambda)=\frac{s(l,\lambda)}{s^{[1]}(l,\lambda)},\quad \gamma_{Dl}(\lambda)=\frac{\mathfrak{s}(\lambda)}{s^{[1]}(l,\lambda)}. \tag{4.16}$$

Theorem 4.4. *The linear relation A_{N0} is symmetric in $L^2(W)$ with deficiency indices $(1,1)$, and the following conditions hold:*

(i) The adjoint linear relation A^*_{N0} has the form

$$A^*_{N0} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(l) = 0 \right\}. \tag{4.17}$$

(ii) The tuple $\{\mathbb{C}, \Gamma_0^{Dl}, \Gamma_1^{Dl}\}$, where

$$\Gamma_0^{N0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(0), \quad \Gamma_1^{N0} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = -f(0), \tag{4.18}$$

is a boundary triple for A^*_{N0}.

(iii) The corresponding Weyl function and the γ-field have the form

$$M_{N0}(\lambda) = \frac{s^{[1]}(l,\lambda)}{c^{[1]}(l,\lambda)}, \quad \gamma_{N0}(\lambda) = \mathfrak{s}(\lambda) - \frac{s^{[1]}(l,\lambda)}{c^{[1]}(l,\lambda)} c(\cdot,\lambda). \tag{4.19}$$

Theorem 4.5. *The linear relation A_{Nl} is symmetric in $L^2(W)$ with deficiency indices $(1,1)$, and the following conditions hold:*

(i) The adjoint linear relation A^*_{Nl} has the form

$$A^*_{N1} = \left\{ \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in A_{max} : f^{[1]}(0) = 0 \right\}. \tag{4.20}$$

(ii) The tuple $\{\mathbb{C}, \Gamma_0^{Nl}, \Gamma_1^{Nl}\}$, where

$$\Gamma_0^{Nl} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f^{[1]}(l), \quad \Gamma_1^{Nl} \begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} = f(l), \tag{4.21}$$

is a boundary triple for A^*_{Nl}.

(iii) The corresponding Weyl function and the γ-field have the form

$$M_{Nl}(\lambda) = \frac{c(l,\lambda)}{c^{[1]}(l,\lambda)}, \quad \gamma_{Nl}(\lambda) = \frac{c(\cdot,\lambda)}{c^{[1]}(l,\lambda)}. \tag{4.22}$$

Remark 4.6. The Weyl functions M_{D0}, M_{N0} in the case $dQ \equiv 0$ coincide with the functions Ω_0, Ω_1, see [18, p. 666, (2.40–41)].

5. Special cases

5.1. Absolutely continuous case. Sturm–Liouville operator

Let the functions P, Q, and W be absolutely continuous on $[0,l]$, i.e., there exist functions p, q, and w from $L^1[0,l]$ such that

$$P(x) = \int_0^x p(t)dt, \quad Q(x) = \int_0^x q(t)dt, \quad W(t) = \int_0^x w(t)dt, \tag{5.1}$$

$p(t) \neq 0$ and $w(t) \geq 0$ almost everywhere with respect to the Lebesgue measure on $[0, l]$. In addition, we require that the space $L^2(W)$ be nontrivial. The last requirement is equivalent to $W(l) > W(0)$.

In this special case, system (1.1) can be written in the form

$$J\vec{f}'(x) = \lambda H(x)\vec{f}(x) + V(x)\vec{f}(x), \quad \vec{f}(0) = \vec{a}(0), \tag{5.2}$$

where

$$H(x) = \begin{pmatrix} w(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} -q(x) & 0 \\ 0 & p(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix}$$

or, equivalently,

$$\begin{cases} -(f^{[1]})'(x) = \lambda w(x)f(x) - q(x)f(x), \\ f'(x) = p(x)f^{[1]}(x). \end{cases} \tag{5.3}$$

System (5.3) is equivalent to the Sturm–Liouville equation (see [26])

$$-\frac{d}{dx}\left(\frac{1}{p(x)}\frac{d}{dx}f(x)\right) + q(x)f(x) = \lambda w(x)f(x). \tag{5.4}$$

with the initial conditions

$$f(0) = a_1, \quad f^{[1]}(0) = a_2.$$

The more general canonical systems (5.2) were studied in [16, 20, 24], where, in particular, it was shown that the maximal and minimal operators associated with such canonical systems can be linear relations with nontrivial multivalued part. In the two-dimensional case, the multivalued part of the maximal operator was calculated explicitly in terms of the so-called H-indivisible intervals [16]. Actually in the absolutely continuous case, the results of the work can be easily derived from the results of [4] and [23].

5.2. Discrete case. Stieltjes string

Let us consider system (3.4) in the case $dQ \equiv 0$, $dP = dx$, and let W be a left-continuous monotonically nondecreasing piecewise constant function on $[0, l]$ that has at least two growth points. Let $\{x_j\}_{j=0}^{n-1}$ be the growth points of W such that

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n := l. \tag{5.5}$$

By w_j, we denote

$$w_j := W(x_j + 0) - W(x_j) \quad (j \in \{1, 2, \ldots, n-1\}). \tag{5.6}$$

The distance between the neighboring growth points is denoted by

$$l_j := x_j - x_{j-1} \quad (j \in \{1, 2, \ldots, n\}). \tag{5.7}$$

Finally for convenience, we denote

$$f_j := f(x_j), \quad g_j := g(x_j), \quad h_j := f^{[1]}(x_j). \tag{5.8}$$

With generating function W, the space $L^2(W)$ is isomorphic to $\mathbb{C}^n$, and every of its elements is a vector $[f_0, f_1, \ldots, f_{n-1}]^T$.

It is easy to verify that, by the above assumptions, one can choose intervals i_1, i_2 closed on the left and such that they satisfy Proposition 3.7. For instance, it is sufficient to choose intervals i_j with the only growth point x_{j-1} ($j \in \{1,2\}$). Then the spaces $L^2(i_j,W)$ obviously are non-trivial, and inequality (3.16) takes the form

$$\frac{w_1 l_1}{w_2} > 0. \tag{5.9}$$

Combining (5.6), (5.7), and (5.8), we can rewrite system (3.4) as

$$\begin{cases} f_{j+1} - f_j = h_{j+1} l_{j+1}, \\ h_{j+1} - h_j = -w_j g_j, \end{cases} \tag{5.10}$$

where $j \in \{0, 1, \dots, n-1\}$.

Proposition 5.1. *In the assumptions of case 5.2, the multivalued part of the linear relation A_{max} has the form*

$$\{(c_1, 0, 0, \dots, 0, c_2)^T : c_1, c_2 \in \mathbb{C}\}, \tag{5.11}$$

and the linear relation A_{min} is the graph of a single-valued linear operator.

Proof. Let $\mathfrak{f}$ be the zero element of $L^2(W)$. Thus, in (5.10), we have $f_j = 0$ as $j \in \{0, 1, \dots, n-1\}$. Hence, $h_j = 0$ as $j \in \{1, 2, \dots, n-1\}$, and $g_j = 0$ as $j \in \{1, 2, \dots, n-2\}$. The converse is also true: since we can choose $h_0 = w_0 g_0$, the pair $\begin{pmatrix} 0 \\ \mathfrak{g} \end{pmatrix}$ belongs to the linear relation A_{max} for each vector $\mathfrak{g} \in L^2(W)$ of the form (5.11).

If, in addition, $f_n = h_0 = h_n = 0$, then it follows from (5.10) that $g_j = 0$ as $j \in \{0, 1, \dots n-1\}$. □

5.3. Mixed case. Krein–Feller string

A more general case can be obtained, if we suppose that $dQ \equiv 0$, $dP = dx$, and W is an arbitrary mototonically nondecreasing function.

Proposition 3.7 holds, iff W has at least two distinct growth points on $[0, l]$:

$$0 < W(x_0) < W(x_1) \le W(l). \tag{5.12}$$

Now, system (3.4) has the form

$$\begin{cases} f(x) - f(0) = \int\limits_0^x f^{[1]}(t)dt, \\ f^{[1]}(x) - f^{[1]}(0) = -\int\limits_0^x g(t)dW(t). \end{cases} \tag{5.13}$$

In particular, we have the following assertion.

Proposition 5.2. *Suppose that the assumptions of case 5.3 are satisfied. If a pair $\begin{pmatrix} \mathfrak{f} \\ \mathfrak{g} \end{pmatrix} \in L^2(W)^2$ belongs to the linear relation A_{max}, then there exists $f \in \mathfrak{f}$ such that f is absolutely continuous with respect to the Lebesgue measure, and its derivative coincides with $f^{[1]}$ almost everywhere.*

Let us rewrite system (5.13) as

$$f(x) = f(0) + x f^{[1]}(0) - \int\limits_0^x \left(\int\limits_0^t g(s)dW(s) \right) dt. \tag{5.14}$$

The function $\int\limits_0^t g(s)dW(s)$ is left-continuous and of bounded variation on $[0,l]$. It follows from (2.14) that equality (5.14) can be rewritten as

$$f(x) = f(0) + xf^{[1]}(0) - \int\limits_0^x (x-s)g(s)dW(s). \tag{5.15}$$

Definition 5.3. [17] A function f is said to belong to the Stieltjes class S^+, if $f \in R$, and f admits a holomorphic non-negative continuation to $(-\infty, 0)$.

In work [18], the differential operation defined by (5.15) was investigated. I. S. Kats and M. G. Krein showed that, under the assumptions of case 5.2, the Weyl functions M_{D0}, M_{N0} and M_{Nl} constructed in Section 4 belong to the Stieltjes class S^+ (see [18, p. 666, Lemma 2.3]).

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